

Some Result of Fuzzy Separation Axiom in Fuzzy Topological ring Space

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Abstract: In this paper, we study fuzzy separation axiom T_i , $i = 0,1,2,3$ in the fuzzy topological ring space. Also the relationship between the types of fuzzy separation axiom was studied.

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1.Introduction:

In 1965 [11], Zadeh L. A. gave the definition of fuzziness. After three years C. Chang [2] gave the notion of fuzzy topology. In 1990[1], Ahsanullah and Ganguli, depended on the convergent in fuzzy topological space in the sense of Lowen[7, 8] to introduce the concept of fuzzy nbhd rings which gives the necessary and sufficient condition for a prefilter basis to be fuzzy nbhd prefilter of 0 in fuzzy topological ring. Also they are study the notions of right and left bounded fuzzy set and precompact fuzzy set fuzzy nbhd rings.

In 2009, Deb Ray, A. and Chettri, P [3] introduced fuzzy topology on a ring. Also in [4] they introduced fuzzy continuous function and studied left fuzzy topological ring. Our working to study fuzzy separation axiom T_i , $i = 0,1,2,3$ in the fuzzy topological ring space and obtaining the relationship between T_i , $i = 0,1,2,3$ spaces in the fuzzy topological ring space

For rich the paper, some basic concept of fuzzy set , fuzzy topology and fuzzy topological ring are given below. The symbol I will denote to the closed interval $[0,1]$.

Definition [11] 1.1

A fuzzy set in R is a map $\vartheta: R \rightarrow I$ and, that is, belonging to I^R (the set of all fuzzy set of R). Let $E \in I^R$, for every $r \in R$, we expressed by $E(r)$ of the degree of membership of r in R . If $E(r)$ be an element of $\{0, 1\}$, then E is said a crisp set.

Definition [2] 1.2

A class $\mu \in I^R$ of fuzzy set is called a fuzzy topology for R if the following are satisfied

- 1) $\emptyset, R \in \mu$
- 2) $\forall E, H \in \mu \rightarrow E \wedge H \in \mu$
- 3) $\forall (E_j)_{j \in J} \in \mu \rightarrow \bigvee_{j \in J} E_j \in \mu$

(R, μ) is called fuzzy topological space. if $A \in \mu$ Then A is fuzzy open and A^c (complement of A) is a fuzzy closed set.

Definition [1, 3] 1.3

A pair (R, μ) , where R a ring and μ be a fuzzy topology on R , is called fuzzy topological ring if the following maps are fuzzy continuous:

- 1) $R \times R \rightarrow R, (r, k) \rightarrow r + k.$
- 2) $R \rightarrow R, r \rightarrow -r$
- 3) $R \times R \rightarrow R, (r, k) \rightarrow r.k$

Definition [4]1.4

A family B of fuzzy nbhds of r_α , for $0 < \alpha \leq 1$, is called a fund. system of fuzzy nbhds

of r_α iff for any fuzzy nbhd V of r_α , there is $U \in B$ such that $r_\alpha \leq U \leq V$

Definition [4]1.5

Let R be a ring and μ a fuzzy topological on R . Let U and V are fuzzy sets in R . We define $U + V, -V$ and $U.V$ as follows

$$(U + V)(k) = \sup_{k=k_1+k_2} \min \{U(k_1), V(k_2)\}$$

$$-V(k) = V(-k)$$

$$(U.V)(k) = \sup_{k=k_1+k_2} \min \{U(k_1), V(k_2)\}$$

Theorem [4]1.6

If R is a fuzzy topological ring then there is a fundamental system of fuzzy nbhds B of 0 ($0 < \alpha \leq 1$), such that the conditions:

- (i) $\forall U \in B$, then $-U \in B$
- (ii) $\forall U \in B$, then U is symmetric
- (iii) $\forall U, V \in B$, then $U \wedge V \in B$
- (iv) $\forall U \in B$, there is $V \in B$ such that $V + V \leq U$
- (v) $\forall U \in B$, there is $V \in B$ such that $V.V \leq U$
- (vi) $\forall r \in R, \forall U \in B$, there is $V \in B$ such that a $r.V \leq U$ and $V.r \leq U$.

Definition [7] 1.7

(R, μ) is fully stratified fuzzy topology on R if the fuzzy topology μ on R contain all constant fuzzy set

Definition1.8 [10]

A fuzzy topological space (R, μ) is said to be fuzzy T_0 -topological space iff $\forall r_\alpha, k_\alpha \in R, r \neq k, \exists U \in \mu$ such that either $U(r) = 1$ and $U(k) = 0$ or $U(k) = 1$ and $U(r) = 0$.

Definition1.9 [10]

A fuzzy topological space (R, μ) is said to be fuzzy T_1 - topological space iff $\forall r_\alpha, k_\alpha \in R, r \neq k, \exists U, V \in \mu$ such that $U(r) = 1, U(k) = 0$ and $V(r) = 0, V(k) = 1$

Definition 1.11[10]

A fuzzy topological space (R, μ) is said to be fuzzy Hausdorff or fuzzy T_2 space iff for any two distinct fuzzy points $r_\alpha, k_\alpha \in R$, there exists disjoint fuzzy sets $U, V \in \mu$ with

$$U(r) = V(k) = 1.$$

Definition 1.12 [10]

A fuzzy topological space (R, μ) will be called fuzzy regular if for each fuzzy point r_α and each fuzzy closed set H such that $H(r) = 0$ there are fuzzy open sets U and V such that $U(r) > 0, H \leq V$ and $U \wedge V = \emptyset$.

Proposition 1.14[10]

If a space R is a fuzzy regular space, then for any open set U and a fuzzy point $r_\alpha \in R$ such that $cl(U)(r) = 0$ there exists an open set V such that $\alpha \leq V \leq cl(V) \leq U$

Definition 1.15[10]

A fuzzy topological space (R, μ) will be called normal if for each pair of

fuzzy closed sets H_1 and H_2 such that $H_1 \wedge H_2 = \emptyset$ there exist fuzzy open sets U_1 and U_2 such that $U_1 \leq H_1$ and $U_2 \leq H_2$ and $U_1 \wedge U_2 = \emptyset$.

Definition 1.16[10]

A fuzzy topological space (R, μ) is said to be fuzzy T_3 -topological space iff it is fuzzy T_1 -and fuzzy regular.

2.Separation Axiom

Theorem 2.1

Let (R, μ) be a fuzzy topological ring. If $k_\alpha \in \{r_\alpha : R(r) = \max\{R(h)\}, \forall h \in R\}$ and U is a fuzzy nbhd of 0 , then $k + U$ is a fuzzy nbhd of k such that $(k + U)(k) = \max\{R(h)\}, \forall h_\alpha \in R\}$.

Proof

Since U is a fuzzy nbhd of 0 , there exists V fuzzy open set such that $V \subseteq U$ and $V(0) = U(0) = \max\{R(h)\}, \forall h \in R\}$. Let $g_k(r_\alpha): (R, \mu) \rightarrow (R, \mu), g_k(r) = r_\alpha + k_\alpha$. g_k is a fuzzy homeo. Thus $k + V$ is a fuzzy open set.

$$\begin{aligned} k + V(k) &= V(k - k) = V(0) \\ &= \max\{R(h)\}, \quad \forall h \in R. \\ k + U(r) &= U(r - k) \geq V(r - k) = k + V(r) \text{ for all } r \in R. \end{aligned}$$

Thus there exists $k + V$ fuzzy open such that $k + V \leq k + U$ and $(k + V)(K) = (k + U)(k) = \max\{R(h)\}, \forall h \in R$

Theorem 2.2

Let (R, μ) be a fuzzy topological ring. If $k_\alpha \in \{r_\alpha : R(r) = \max\{R(h)\}, \forall h \in R\}$ and U is a fuzzy nbhd of k_α such that $U(k) = \max\{R(h)\}, \forall h \in R$, then $U - k$ is a fuzzy nbhd of 0 such that $(U)(0) = \max\{R(h)\}, \forall h \in R$.

Proof

Since U is a fuzzy nbhd of k_α , there exists V fuzzy open set such that $V \subseteq U$ and $V(k) = U(k) = \max\{R(h)\}, \forall h \in R$. Let $g_k: R \rightarrow R, g_k(r_\alpha) = r_\alpha - k_\alpha, g_k$ is a fuzzy homeo. Thus $-k + V$ is a fuzzy open set.

$$V - k(0) = V(0 + k) = V(k) = \max\{R(h)\}, \forall h \in R\} = V(0).$$

$$U - k(r) = U(r + k) \geq V(r + k) = V - k(r)$$

for all $r \in R$. Thus there exists $V - k$ fuzzy open set such that $V - k \leq U - k$ and $V(0) = U - k(0) = \max\{R(h)\}, \forall h \in R\}$.

Definition 2.3

A fuzzy topological ring (R, μ) is said to be Fuzzy T_0 - space iff for each r_α, k_α s.t $r \neq k$ there exists fuzzy open set U s.t $r_\alpha \in U, k_\alpha \notin U$ or $k_\alpha \in U, r_\alpha \notin U$

Example 2.4

Let Z_2 be the ring of integers modulo 2. Define fuzzy sets E_1, E_2, E_3 on Z_2 as $E_1([0]) = 0.9, E_1([1]) = 0, E_2([0]) = 0, E_2([1]) = 0.9$

for all $r \in Z_2$. Let $\mu = \{\emptyset, Z_2, E_1, E_2\}$ is a fuzzy topological ring on Z_2 , then (Z_2, μ) is a fuzzy T_0 - topological ring space.

Definition 2.5

A fuzzy topological ring (R, μ) is said to be Fuzzy T_1 - space iff for each r_α, k_α s.t $r \neq k$ there exists fuzzy open sets U, V s.t $r_\alpha \in U, k_\alpha \notin U$ and $k_\alpha \in V, r_\alpha \notin V$

Example 2.6

Let Z_2 be the ring of integers modulo 2. Define fuzzy sets E_1, E_2, E_3 on Z_2 as $E_1([0]) = 0.25, E_1([1]) = 0, E_2([0]) = 0, E_2([1]) = 0.25, E_3([0]) = 0.25, E_3([1]) = 0.25$

for all $r \in Z_2$. Let $\mu = \{\emptyset, Z_2, E_1, E_2, E_3\}$ is a fuzzy topological ring on Z_2 , then (Z_2, μ) is a fuzzy T_1 - topological ring space.

Definition 2.7

A fuzzy topological ring (R, μ) is said to be Fuzzy T_2 - topological ring space iff for any two fuzzy points r_α, k_β s.t $r \notin \text{supp}(k)$ and $k \notin \text{supp}(r)$, there exists two fuzzy open sets U, V s.t $r_\alpha \in U, k_\alpha \in V$ and $U \wedge V = \emptyset$

Example 2.8

Let Z_4 be the ring of integers modulo 4, with fuzzy discrete topology μ_D on it. The fuzzy topological ring (Z_4, μ_D) is Fuzzy T_2 - topological ring space.

Definition 2.9

A fuzzy topological ring (R, μ) is said to be Fuzzy regular space if $\forall r_\alpha \in R$ and a fuzzy closed set F with $F(r) = 0$ there exists fuzzy open sets U, V such that $r_\alpha \in U$ and $F \subset V$ and $\Lambda V = \emptyset$.

Definition 2.10

A fuzzy topological ring (R, μ) is said to be Fuzzy T_3 –topological ring space if (R, μ) is fuzzy T_1 – space and fuzzy regular topological ring space.

Example 2.11

Let \mathbb{R} be the ring of real number, with fuzzy usual topology μ_U on it. Then (\mathbb{R}, μ_U) is Fuzzy T_3 – topological ring space

Theorem 2.12

If (R, μ) is fuzzy T_2 - topological ring space then $\{0_\alpha\}$ is fuzzy closed subset in (R, μ) .

Proof

Let (R, μ) is fuzzy T_2 - topological ring space. For any $r_\alpha \neq 0_\alpha$ be another fuzzy point, assume $U(r) > 0, \forall U \in \{B_0\}$, then there exists fuzzy open set V of r_α s.t $V(0) = 0$, implies $\overline{\{0\}}(r) = 0$. Since r is arbitrary. Thus $\overline{\{0_\alpha\}} = \{0_\alpha\}$ and $\{0_\alpha\}$ is fuzzy closed set.

Theorem 2.13

For any fuzzy topological ring space (R, μ) , if $\{0_\alpha\}$ is fuzzy closed subset in R and if B_0 is a basis of fuzzy nbhd of 0_α , then $\Lambda_{V \in \{B_0\}} V = \{0_\alpha\}$.

proof

Let $\overline{\{0_\alpha\}}$ be an fuzzy closed set and $\{B_0\}$ be a basis of fuzzy nbhds of 0_α . Then by theorem 2.12, $\{0_\alpha\} = \overline{\{0_\alpha\}} = \Lambda_{V \in \{B_0\}} (\{0_\alpha\} + V) = \Lambda_{V \in \{B_0\}} V$
Thus $\Lambda_{V \in \{B_0\}} V = \{0_\alpha\}$

Theorem 2.14

For any fuzzy topological ring space (R, μ) , and B_0 is a basis of fuzzy nbhd of 0_α , if $\Lambda_{V \in \{B_0\}} V = \{0_\alpha\}$, then (R, μ) is fuzzy T_0 -topological ring space.

Proof

Let $\{B_0\}$ be a basis of fuzzy nbhds of 0_α and $\Lambda_{V \in \{B_0\}} V = \{0_\alpha\}$. Let r_α and k_α are fuzzy points with different support. Therefor $r_\alpha - k_\alpha \neq 0$, so there exists $V \in \{B_0\}$, s.t $V(r - k) = 0$. Now by theorem 1.6, $k + V(r)$ is fuzzy nbhd of k_α and $(k + V)(r) = V(r - k) = 0$. Thus $r_\alpha \notin k + V$ and (R, μ) is fuzzy T_0 -topological ring space.

Theorem 2.15

If (R, μ) is fuzzy T_0 - topological ring space, Then (R, μ) is fuzzy T_1 -topological ring space

Proof

Let (R, μ) be a fuzzy T_0 -topological ring space and let r_α and k_α are fuzzy points with

different support. Then there exists fuzzy open set V of 0_α s.t $(r + V)(k) = 0$ or $(k + V)(r) = 0$. We can assume that V is symmetric fuzzy open set of 0_α . To explain that $(r + V)(k) = 0$ if $(k + V)(r) = 0$ we assume the contrary, suppose that $(r + V)(k) > 0$, therefor $(r - V)(k) > 0$.
Implies

$$(r + V)(k) = V(k - r) = V(-(r - k)) = V(r - K) = (k + V)(r) > 0.$$

This contradiction, similarly if $(k + V)(r) = 0$. Thus (R, μ) is fuzzy T_1 -topological ring space.

Theorem 2.16

If (R, μ) is fuzzy T_1 -topological ring space then (R, μ) is fuzzy T_3 -topological ring space.

Proof

Let (R, μ) is fuzzy T_1 -topological ring space and let $r \in R, F$ be a fuzzy closed subset in R s.t $F(r) = 0$, then $F^c(r) = 1$ and F^c is fuzzy open set. Therefor $F^c - r$ is an fuzzy open nbhd system of 0_α . Then there exists fuzzy open set V_0 of 0_α s.t $V_0 \leq F^c - r$. Now

$$\bar{V}_0 = \cap(V_0 + (F^c - r)) = \cap(F^c - r) = \{0_\alpha\},$$

$$(r + \bar{V}_0)(k) = \bar{V}_0(k - r) = \min\{\bar{V}_0(k), \bar{V}_0(-r)\} = \min\{\bar{V}_0(k), \bar{V}_0(r)\} = \bar{V}_0(k) = V_0(k) \quad , \forall k \in R$$

implies $r + \bar{V}_0 \leq F^c - r$. Thus (R, μ) is fuzzy regular and consequently it is fuzzy T_3 -topological ring space.

Theorem 2.17

For any fuzzy topological ring (R, μ) , the following conditions are equivalent

- 1) (R, μ) is fuzzy T_2 -topological ring space
- 2) $\{0_\alpha\}$ is FZ.closed subset in R .
- 3) If B_0 is a basis of nbhd of 0_α , then $\cap_{V \in B_0} V = \{0_\alpha\}$
- 4) (R, μ) is fuzzy T_0 -topological ring space.
- 5) (R, μ) is fuzzy T_1 -topological ring space.
- 6) (R, μ) is fuzzy T_3 -topological ring space.

Proof

By theorems 2.10, 2.11, 2.12, 2.13 and 2.14

Theorem 2.18

Let (R, μ) be a fuzzy topological ring, then Every fuzzy subspace of fuzzy T_0 -space is a fuzzy T_0 - space.

Proof:

Obvious

Theorem 2.19

Let (R, μ) be a fuzzy topological ring and E is a fuzzy subring of (R, μ) , if (E, μ_E) is fuzzy Hausdorff and $(R/E, \beta)$ is fuzzy Hausdorff then (R, μ) is fuzzy Hausdorff.

Proof

If R/E is fuzzy Hausdorff, then E is fuzzy closed set in (R, μ) .

If also (E, μ_E) is fuzzy Hausdorff then $\{0_\alpha\}$ is fuzzy closed in E .

Hence $\{0_\alpha\}$ is fuzzy closed in (R, μ) , and (R, μ) is fuzzy Hausdorff.

References

- [1] Ahsanullah T. M. G., On Fuzzy Neighborhood Ring, Fuzzy Set and Systems 34(1990) 255-262 North Holland
- [2] Chang, C. L : Fuzzy topological spaces. Math. Anal. Appl.,24(1968),182-190.
- [3] Deb Ray, A and Chettri, P: On Fuzzy Topological Ring Valued Fuzzy Continuous Functions "Applied Mathematical Sciences, Vol. 3, 2009, no. 24, 1177 – 1188
- [4] Deb Ray, A : On (left) fuzzy topological ring. Int. Math. vol. 6 (2011), no. 25 – 28, 1303 – 1312.
- [5] Das, N.R. and Das, P. : Neighborhood systems in fuzzy topological groups. Fuzzy Sets and Systems, 116 (2000) 401-408
- [6] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl, 56(1976) 621-633
- [7] Lowen R., Convergence in a fuzzy topological space, Gen. Topology Appl. 10 (1979)147-160
- [8] Lowen R., Fuzzy neighborhood spaces, J. Fuzzy Sets and Systems 7(1982), 165-189

[9] Palaniappan N., Fuzzy topology, Narosa Publications, 2002.

[10] Warner S.: Topological Rings, North-Holland Math. 1993

[11] Zadeh, L.A : Fuzzy Sets, Information and Control,8(1965), 338-353.